Some notes on quantum foundations

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Note: This is a document that exists primarily for my own use, as a place to record and clarify my understanding of some foundational issues in quantum theory. I intend to refine and extend it over time. To that end, I would be pleased if it is useful to anyone else, and even more pleased if anyone has any comments on (or vehement disagreements with) anything I've written. Let me know!

1 Questions raised by quantum mechanics

Textbook quantum mechanics is based on the following postulates:

1. Each physical system S is associated to a separable complex Hilbert space $\mathcal{H}_S$.
2. If a system S is composed of two disjoint subsystems A and B, then $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$.
3. The state of an isolated system is given by a normalized vector $\psi \in \mathcal{H}_S$.
4. The state of an isolated system evolves unitarily in time.
5. An agent may choose to measure a system S. This action involves a choice of closed subspace $V \subseteq \mathcal{H}_S$, followed by either (a) the agent receiving the answer "yes" with probability $p = \langle \psi, \Pi_V \psi \rangle$ and the state is projected onto V and renormalized, or (b) the agent receiving the answer "no" with probability $1-p$ and the state is projected onto $V^{\perp}$ and renormalized.

This framework immediately invites several questions, including at least the following five:

Q1: Does this model actually capture all the phenomena we know of?

Certainly it is not manifestly relativistically covariant, as the processes of measurement and unitary evolution are described in a preferred reference frame. In addition, incorporating bosonic and fermionic statistics would require some extra structure. Destructive measurements, such as photon polarization detection are also not supported by the notion of a static set of systems.

Q2: Why should we assign a Hilbert space to each system and compute probabilities using the Born rule and update the state with unitaries?

If we were to write down a classical theory modeled on this structure, we would assign a set to each system, with the elements of the set nothing more than the possible states of that system. This seems like a plausible thing to do from a purely *a priori* point of view: of course a physical theory should use in its mathematical expression the collection of all of the ways that things can be. What does the extra structure mean?

Q3: Are probabilistic outcomes fundamental, or can we convert obtain a deterministic theory somehow?

This is the problem of hidden variables. It would be nice if it turned out that a quantum state simply corresponded to a probability distribution over a state space of the classical form discussed in the previous question. It would then become interesting to ask why there seems always to be this particular kind of restriction on what we can know about the state of a system, but our classical ontological intuition would be rescued.

Q4: Are the system/agent and unitary/measurement dichotomies fundamental, or can we dispose of this distinction?

This is the measurement problem. In principle there is no reason that these distinctions could not be fundamental (the theory works, after all, and is mathematically consistent), but it would be very strange. In classical mechanics, there is no obstruction to thinking about an agent as a physical system in its own right, but examples such as the Wigner's friend thought experiment demonstrate the difficulty in treating agents as quantum systems.

Q5: Can the agents do anything interesting that they couldn't if the systems were classical?

As a yes or no question, this one is easy: yes. Quantum cryptography, communication, and computation (at least with oracles) provide examples of protocols that are possible with quantum mechanics and not classical, and there is presumably much more to discover. This sort of analysis, where the quantum nature of the physical systems is treated as a resource for accomplishing informational tasks, might be useful for thinking about the prior three questions.

2 Quantum Logic

The goal of this note is to discuss what quantum theory has to say about the structure of the set of propositions about a physical system, a program initiated by Birkhoff and von Neumann in [2]. In order to do so, we first need to be clear about what these propositions actually are. We will focus on the following fragment of textbook quantum mechanics:

Quantum mechanics, for our purposes

Each isolated physical system is associated to a complex Hilbert space $\mathcal{H}$, which we'll take to be finite-dimensional. The state of the system is a norm-1 vector $\psi \in \mathcal{N}$, where $\mathcal{N}$ denotes the unit sphere of $\mathcal{H}$. Each two-outcome measurement is associated to a subspace $V \subseteq \mathcal{H}$. If the system is in the state ψ, the probability of receiving the answer `yes` upon performing the measurement associated to the subspace V is given by the Born rule.

Propositions about a quantum system

We are now in a position to think about propositions that some user of quantum mechanics could consider. There are multiple types of propositions about a physical system suggested by the formalism described in the last paragraph:

$$(S) = \text{“The state } \psi \text{ of the system is in the subset } S \subseteq \mathcal{N}(\mathcal{H})\text{”.} \tag{1}$$
$$[V] = \text{“The measurement corresponding to the subspace } V \subseteq \mathcal{H} \text{ would yield the answer } \texttt{yes}\text{.”} \tag{2}$$
$$[V]' = \text{“The measurement corresponding to the subspace } V \subseteq \mathcal{H} \text{ would yield the answer } \texttt{no}\text{.”} \tag{3}$$

Denote by $\mathcal{P}$ the set of all such propositions. For each proposition, a user of quantum mechanics may hold one of (at least) three attitudes: belief, rejection, or uncertainty. For some proposition $x \in \mathcal{P}$, we will denote these attitudes by $\phi(x) = 1$, 0, or $-$, respectively, so that $\phi : \mathcal{P} \to \{1, 0, -\}$ is an object that encodes the user's attitudes towards all propositions about the system. Quantum mechanics (along with set theory) places restrictions of logical coherence on ϕ. For two propositions $x, y \in \mathcal{P}$, we will say that x implies y, denoted $x \to y$, if these restrictions require that if $\phi(x) = 1$ then $\phi(y) = 1$, and if $\phi(y) = 0$ then $\phi(x) = 0$. In particular, we see that the following hold for any subsets $S, T \subseteq \mathcal{N}$ and any subspace $V \subseteq \mathcal{H}$:

$$(S) \to (T) \qquad \text{iff} \qquad S \subseteq T \tag{4}$$
$$(V \cap \mathcal{N}) \leftrightarrow [V] \tag{5}$$
$$\left(V^{\perp} \cap \mathcal{N}\right) \leftrightarrow [V]', \tag{6}$$

where $V^{\perp}$ is the orthogonal complement of V.

Definition 1. *A poset $(P, \to)$ is a set P together with a reflexive, transitive binary relation $\to$ such that $x \to y$ and $y \to x$ implies $x = y$ (antisymmetry).*

If we consider only the propositions of the form (S), equipped with the restriction of the relation $\to$, we obtain a poset we will call $\mathcal{P}_{\text{state}}$. This is nothing but the powerset of $\mathcal{N}$, partially ordered by inclusion. If we consider only propositions about measurement outcomes, i.e. propositions of the form $[V]$ or $[V]'$, we can identify the logically equivalent propositions $[V]'$ and $\left[V^{\perp}\right]$ to obtain the poset

$$\mathcal{P}_{\text{meas}} = \{[V] : V \subseteq \mathcal{H} \text{ a subspace}\} \tag{7}$$

partially ordered by the subspace relation.

From classical physics we have the intuition that, given enough information, we can determine the state of a system to arbitrary precision, and that a measurement is merely the revelation of a certain state-dependent property of the system (presumably with some continuity conditions). Then an agent with enough information should in principle be able to assign either 0 or 1 to all propositions of either the form (S) or the form $[V]$. This intuition fails in quantum mechanics. In principle, an agent can assign truth values simultaneously to all of the propositions (S), but because of the existence of non-commuting projectors on $\mathcal{H}$, no agent can do this for the propositions $[V]$.

Lattices of propositions

Definition 2. *A lattice $(L, \to)$ is a poset $(L, \to)$ such that for any two elements $x, y \in L$ there exist both (1) a least upper bound or join $x \vee y \in L$ such that for any $z \in L$ if $x \to z$ and $y \to z$ then $x \vee y \to z$ and (2) a greatest lower bound or meet $x \wedge y \in L$ such that if $z \to x$ and $z \to y$ then $z \to x \wedge y$.*

In fact, both of the posets defined above are lattices, with meets and joins given by

$$(S) \wedge (T) = (S \cap T) \tag{8}$$
$$(S) \vee (T) = (S \cup T) \tag{9}$$
$$[V] \wedge [W] = [V \cap W] \tag{10}$$
$$[V] \vee [W] = [V \oplus W] \tag{11}$$

where $V \oplus W$ denotes the linear span of the subspaces V and W, i.e. the smallest subspace containing both. Note that these equalities are *not* definitions, but rather may be derived from the definitions of the partial orders on $\mathcal{P}_{\text{state}}$ and $\mathcal{P}_{\text{meas}}$. The following definitions of lattice-theoretic terms are taken from [4].

Definition 3. *A zero or minimum of a lattice is an element 0 such that $0 < x$ for all x. A unit or maximum is an element 1 such that $x < 1$ for all x. A lattice with both 0 and 1 is said to be bounded.*

Definition 4. *In a bounded lattice, an element y is said to be a complement of the element x if $x \vee y = 1$ and $x \wedge y = 0$. A bounded lattice is said to be complemented if each element has a complement in the lattice.*

Definition 5. *An orthocomplementation on a complemented lattice is a map $'$ that assigns to each element x an element x' such that x' is a complement of x, $(x')' = x$, and if $x < y$ then $y' < x'$. A bounded lattice equipped with an orthocomplementation is called an orthocomplemented lattice or an ortholattice.*

Definition 6. *An ortholattice is said to be orthomodular if $x < z$ implies $x \vee (x^{\perp} \wedge z) = z$.*

Definition 7. *A lattice is complete if any subset of the lattice has a meet and a join.*

Definition 8. *An element a is said to cover an element b, denoted $a \succ b$, if $a \neq b$, $b \to a$, and for any x such that $b \to x$ and $x \to a$, $x = a$ or $x = b$.*

Definition 9. *A nonzero element a of a lattice with 0 is called an atom if $a \succ 0$. A lattice is atomistic if every element x is the join of the set of atoms a such that $a \to x$.*

Definition 10. *A lattice with 0 is said to satisfy the covering law if for any x and any atom a, $a \wedge x = 0$ implies $(a \vee x) \succ x$.*

Definition 11. *A propositional system is a complete, atomistic, orthomodular lattice satisfying the covering law, i.e. for any $x \in L$ and any atom $a \in L$, $a \wedge x = 0$ implies that $a \vee x$ covers x [5].*

Definition 12. *A lattice is said to be distributive if for all x, y, z we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*

Definition 13. *A Boolean lattice is a distributive complemented lattice.*

Both $\mathcal{P}_{\text{state}}$ and $\mathcal{P}_{\text{meas}}$ are propositional systems, and $\mathcal{P}_{\text{state}}$ is Boolean. In $\mathcal{P}_{\text{state}}$, we have

$$0 = (\varnothing) \qquad 1 = (\mathcal{H}) \qquad (S)' = (S^c) \tag{12}$$

and in $\mathcal{P}_{\text{meas}}$, we have

$$0 = [\mathbf{0}] \qquad 1 = [\mathcal{H}] \qquad [V]' = \left[V^{\perp}\right]. \tag{13}$$

Quantum `AND`, `OR`, `NOT`?

A question that suggests itself: To what extent may we interpret the operations $\wedge$, $\vee$, and $'$ in the lattice $\mathcal{P}_{\text{meas}}$ of "quantum propositions" as the `AND`, `OR`, and `NOT` of classical Boolean logic?

The classical Boolean logical operations `AND`, `OR`, and `NOT` should not be thought of as acting on propositions of the form $[V] \in \mathcal{P}_{\text{meas}}$. Rather, they may be used to generate propositions from the elementary set

$$A(V, \phi) = \text{"The agent has attitude } \phi \text{ about the proposition } [V]\text{"}. \tag{14}$$

A given belief descriptor ϕ then determines a truth value, `true` or `false`, for all of these propositions by assigning true to a basic proposition if $\phi = \phi([V])$ and false otherwise, and using the standard truth tables to assign values to compound propositions. `AND` and the meet operation may be thought of synonymously:

$$A([V], 1) \texttt{ AND } A([W], 1) \qquad = \qquad A([V] \wedge [W], 1). \tag{15}$$

However, `OR` does not line up neatly with the join operation. Suppose that V and W are orthogonal subspaces and the agents beliefs are determined by a state ψ which lies in neither V nor W, but does lie in their span. Then $A([V], 1)$ and $A([W], 1)$ are assigned `false` while $A([V] \vee [W], 1) = A([V \oplus W], 1)$ is assigned `true` so that

$$A([V], 1) \texttt{ OR } A([W], 1) \qquad \neq \qquad A([V] \vee [W], 1). \tag{16}$$

Therefore, it should not come as too great a surprise that $\mathcal{P}_{\text{meas}}$ fails to be distributive, and more importantly *this failure is in no way a failure of the distributivity of `AND` and `OR`.* Moreover, we have

$$\texttt{NOT } A([V], 1) = A([V], 0) \texttt{ OR } A([V], -) \neq A([V], 0) = A([V]', 1). \tag{17}$$

Note that neither of these non-equalities is in any way paradoxical. Indeed, the `NOT` non-equality would also appear if we repeated this analysis with the (essentially classical) propositions $\mathcal{P}_{\text{state}}$. Any apparent paradox comes from conflating propositions about a system with propositions about an agents beliefs about propositions of a system. The only difference is that in the case of $\mathcal{P}_{\text{meas}}$ we cannot even in principle restrict to belief descriptors assigning every proposition 1 or 0, which would allow us to collapse the two levels of propositions.

Reconstruction of quantum theory

Another question: How much of the structure of quantum mechanics is captured by the lattice $\mathcal{P}_{\mathrm{meas}}$?

Classical physics assigns to each system a set Ω, the state space, and represents the state of the system by an element $\omega \in \Omega$. Each two-outcome measurement is associated to a subset $S \subseteq \Omega$. If the system is in the state ω, the measurement associated to the subset S will yield the answer `yes` if $\omega \in S$, and the answer `no` if $\omega \notin S$. We can in principle define the propositions (S), $[V]$, and $[V]'$ as in the quantum case, but for most purposes this would be needlessly pedantic - the structure of $\mathcal{P}_{\mathrm{state}}$ is exactly that of $\mathcal{P}_{\mathrm{meas}}$.

Entertaining for a moment the distinction between $\mathcal{P}_{\mathrm{state}}$ and $\mathcal{P}_{\mathrm{meas}}$, we may imagine that we would like to reconstruct the framework of classical physics from the lattice $\mathcal{P}_{\mathrm{meas}}$, which is Boolean. In fact we can do so, thanks to Stone's theorem:

Theorem 1. *A lattice is Boolean iff it is isomorphic to a field of sets.*

See e.g. Corollary 21 of Chapter 7 in [4]. Note that a field of sets is a subset $\mathcal{F} \subseteq 2^{\Omega}$ of the powerset of some set Ω such that $\mathcal{F}$ is closed under finite intersections and unions as well as set complement.

This is a nice result (and mathematically apparently quite powerful) but from the point of view of classical physics not terribly useful. After all, it is very natural to start from a set Ω of "ways the world can be" and to assume that measurements reveal values of physical quantities, which are simply functions $\Omega \to \mathbb{R}$. The situation is very different in quantum mechanics. It does not seem at all natural to posit that the state space of a physical system should be a complex Hilbert space. There is thus a motivation for looking for an analogue to Stone's theorem. Ideally, we would like to be able to justify in some satisfactory way that $\mathcal{P}_{\mathrm{meas}}$ should have some structure as an abstract lattice and then apply the "quantum Stone's theorem" to conclude that $\mathcal{P}_{\mathrm{meas}}$ must be isomorphic to the lattice of subspaces of some complex Hilbert space. Taking $\mathcal{P}_{\mathrm{meas}}$ to be some propositional system, we can't quite get there: Piron's theorem (informally) allows us to conclude that $\mathcal{P}_{\mathrm{meas}}$ is isomorphic to the lattice of subspaces of some *generalized* Hilbert space (see [5, 6] for far more detail). Importantly, Piron's theorem does not allow us to conclude that the set of propositions about measurement outcomes of a quantum system should be isomorphic to the lattice of subspaces of a complex Hilbert space.

Soler's theorem

3 Hidden variables, non-locality, and contextuality

Consider a system described by a finite-dimensional Hilbert space $\mathcal{H}$. The set of $\{0,1\}$-valued questions we can ask about the system is exactly the set of subspaces $V \subseteq \mathcal{H}$. Suppose that the system is isolated, and does not undergo any dynamics other than those resulting from measurement. If an agent makes a sequence of measurements $\mathbf{V} = V_1, V_2, \ldots, V_N$, we would like to be able to assign probabilities $P_{\mathbf{V}}(\mathbf{s})$ to each of the possible sequences $\mathbf{s} = s_1, s_2, \ldots, s_N$ of outcomes. There are various ways to do so:

The orthodox way to define these probability distributions is to specify a unit vector $\psi \in \mathcal{H}$ and define the probabilities

$$P_{\mathbf{V}}(\mathbf{s}) = \left\| \Pi_{V_N}^{s_N} \left(\mathbb{1} - \Pi_{V_N}\right)^{1-s_N} \cdots \Pi_{V_1}^{s_1} \left(\mathbb{1} - \Pi_{V_1}\right)^{1-s_1} \psi \right\|^2 . \tag{18}$$

The data associated to this probabilistic model is the pair $(\mathcal{H}, \psi)$.

Another way we might define the probabilities is to assume that there is some state space Ω associated with the system, with the state determining uniquely the outcome to any question that might be asked. The system may be prepared randomly according to some initial probability distribution μ on Ω. In order to compute the probabilities of outcome sequences for a given sequence of measurements, we need to specify the way in which the states determine outcomes and the way that measurements affect the states. We therefore introduce the following objects:

IZ	ZI	ZZ
XI	IX	XX
$-XZ$	$-ZX$	YY

Table 1: Mermin-Peres magic square.

1. $\mathcal{F}$ is a map from subspaces $V \subseteq \mathcal{H}$ to functions $\mathcal{F}_V : \Omega \to \{0, 1\}$. The interpretation of this map is that if V is measured and the system is in the state ω, the answer $\mathcal{F}_V(\omega)$ will be obtained.
2. $\mathcal{R}$ is a map from subspaces $V \subseteq \mathcal{H}$ to functions $\mathcal{R}_V : \Omega \to \Omega$. The interpretation of this map is that if V is measured and the system is in the state ω, the state is updated according to $\omega \mapsto \mathcal{R}_V(\omega)$.

Then we have the probabilities

$$P_{\mathbf{V}}(\mathbf{s}) = \mu \left\{ \left(\mathcal{F}_{V_1}^{-1}(s_1)\right) \cap \left(\mathcal{R}_{V_1}^{-1} \circ \mathcal{F}_{V_2}^{-1}(s_2)\right) \cap \cdots \cap \left(\mathcal{R}_{V_1}^{-1} \circ \mathcal{R}_{V_2}^{-1} \circ \cdots \circ \mathcal{R}_{V_{N-1}}^{-1} \circ \mathcal{F}_{V_N}^{-1}(s_N)\right) \right\}. \tag{19}$$

The data associated to this probabilistic model is the quintuple $(\mathcal{H}, \Omega, \mu, \mathcal{F}, \mathcal{R})$.

If V and W are subspaces whose projectors commute, the corresponding measurements are compatible: if V is measured and an outcome s_V obtained, then W is measured, then V measured again, the outcome will again be s_V. In other words, as long as we are content to measure only compatible subspaces, we may think of the outcomes as revealing pre-existing properties of the system. A way that this could be implemented at the level of the hidden variable is to demand that $\mathcal{F}_W \circ \mathcal{R}_V = \mathcal{F}_W$ whenever $[\Pi_V, \Pi_W] = 0$. Then for any compatible subspaces $V_1, \ldots, V_N$, we have

$$\mathcal{F}_{V_1} \circ \mathcal{R}_{V_2} \circ \cdots \circ \mathcal{R}_{V_N} = \mathcal{F}_{V_1}. \tag{20}$$

so that the probabilities for sequences of outcomes simplify as

$$P_{\mathbf{V}}(\mathbf{s}) = \mu \left\{ \mathcal{F}_{V_1}^{-1}(s_1) \cap \mathcal{F}_{V_2}^{-1}(s_2) \cap \cdots \cap \mathcal{F}_{V_N}^{-1}(s_N) \right\}. \tag{21}$$

Call this requirement *non-contextuality.*

Theorem 2. *Given a Hilbert space $\mathcal{H}$ of dimension at least four, any quantum probabilistic model $(\mathcal{H}, \psi)$ produces probabilities that are not reproduced by any hidden variable model $(\mathcal{H}, \Omega, \mu, \mathcal{F}, \mathcal{R})$.*

Proof. This construction is known as the Mermin-Peres magic square [3].

Take a four-dimensional subspace of $\mathcal{H}$ that contains ψ, and consider the Table 1 of operators on this subspace, with the commutation relations of the two-qubit Pauli operators. Consider sequences of measurements defined by choosing a row or a column of the table and measuring in some order the $+1$-eigenspaces of the three operators. Because each row and each column defines a set of commuting operators, these measurements are compatible. Moreover, the product of the operators in a row is the identity, while the product of the operators in a column is minus the identity. Therefore, the quantum model $(\mathcal{H}, \psi)$ predicts that with probability one a measurement sequence corresponding to a row (column) will yield an odd (even) number of "yes" responses.

Suppose that $(\mathcal{H}, \Omega, \mu, \mathcal{F}, \mathcal{R})$ is a non-contextual hidden-variable model that also makes these predictions. Then (21) implies that, except possibly on a subset of Ω of μ-measure zero, we must have $\mathcal{F}_V(\omega) = 1$ for an odd (even) number of the subspaces determined by the operators in the rows (columns) of the table. But it is impossible to fill in a 3×3 table with ones and zeros in this way. □

Note that another way to get around this problem is to suppose that the evaluation maps $\mathcal{F}_V$ depend on the set of compatible measurements (the context) that an agent plans to measure. In other words, the outcome of a measurement of V depends on which other measurements might be made. Such a model would be a *contextual* hidden variable model.

Another feature we would like to retain from our classical intuition is the independence of separate subsystems. Suppose that a system S is composed of two subsystems A and B, so that $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Denote by V_A the subspace $V \otimes \mathbb{1}$ and by W_B the subspace $\mathbb{1} \otimes W$. A reasonable requirement on the hidden variable theory is that making a measurement of A shouldn't affect the outcome of a measurement on B. In other words, we can consider requiring that $\mathcal{F}_{W_B} \circ \mathcal{R}_{V_A} = \mathcal{F}_{W_B}$. Then if we imagine an experiment in which a measurement on system A is followed by a measurement on system B, the probabilities simplify:

$$P_{V_A,W_B}(s_A, s_B) = \mu\left\{\mathcal{F}_{V_A}^{-1}(s_A) \cap \mathcal{F}_{W_B}^{-1}(s_B)\right\}. \tag{22}$$

Call this requirement *locality.*

Theorem 3. *If $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is a Hilbert space associated to a bipartite system with $\mathcal{H}_A$, $\mathcal{H}_B$ both at least two-dimensional, there is a quantum probabilistic model $(\mathcal{H}, \psi)$ that generates a set of probabilities not reproduced by any local hidden variable model $(\mathcal{H}, \Omega, \mu, \mathcal{F}, \mathcal{R})$.*

Proof. This is Bell's theorem [1].

Take $|0_A\rangle, |1_A\rangle \in \mathcal{H}_A$ orthogonal unit vectors in $\mathcal{H}_A$, and similarly for $\mathcal{H}_B$. Define the state

$$\psi = \frac{1}{\sqrt{2}}\left(|0_A\rangle |1_B\rangle - |1_A\rangle |0_B\rangle\right). \tag{23}$$

Take V to be a one-dimensional subspace of the abstract two-dimensional space $\mathbb{C}^2$ spanned by $|0\rangle$ and $|1\rangle$. Let V_A be the image of V under the map $|0\rangle \mapsto |0_A\rangle \otimes \mathbb{1}_B$, $|1\rangle \mapsto |1_A\rangle \otimes \mathbb{1}_B$, and similarly for V_B. If a measurement of V_A, yielding outcome s_A, is followed by a measurement of W_B, yielding s_B, the quantum model $(\mathcal{H}, \psi)$ predicts that the two outcomes differ with probability one.

Let $(\mathcal{H}, \Omega, \mu, \mathcal{F}, \mathcal{R})$ be a local hidden-variable model. Define

$$f_V(\omega) = (-1)^{\mathcal{F}_V(\omega)}. \tag{24}$$

From (22) we see that in order for the probabilities to agree with those defined by $(\mathcal{H}, \psi)$, we must have $f_{V_A} \equiv -f_{V_B}$ except possibly on a set of μ-measure zero. Then for arbitrary V and W, we have

$$\langle f_{V_A} f_{W_B}\rangle = \int f_{V_A}(\omega) f_{W_B}(\omega)\, d\mu(\omega) = -\int f_{V_A}(\omega) f_{W_A}(\omega)\, d\mu(\omega) \tag{25}$$

so that for any V, W, Z we find

$$\langle f_{V_A} f_{W_B}\rangle - \langle f_{V_A} f_{Z_B}\rangle = \int f_{V_A}(\omega) f_{W_B}(\omega)\, d\mu(\omega) = \int \Big(f_{V_A}(\omega) f_{Z_A}(\omega) - f_{V_A}(\omega) f_{W_A}(\omega)\Big)\, d\mu(\omega) \tag{26}$$

$$= \int f_{V_A}(\omega) f_{Z_A}(\omega)\Big(1 - f_{W_A}(\omega) f_{Z_A}(\omega)\Big)\, d\mu(\omega). \tag{27}$$

Taking the absolute value of both sides, we find

$$|\langle f_{V_A} f_{W_B}\rangle - \langle f_{V_A} f_{Z_B}\rangle| \leq \int |f_{V_A}(\omega) f_{Z_A}(\omega)| \times |1 - f_{W_A}(\omega) f_{Z_A}(\omega)|\, d\mu(\omega) \tag{28}$$

$$= \int \big(1 - f_{W_A}(\omega) f_{Z_A}(\omega)\big)\, d\mu(\omega) = 1 + \langle f_{W_A} f_{Z_B}\rangle. \tag{29}$$

If we define $p(V, W)$ to be the probability that a measurement of V on system A followed by a measurement of W on system B yield the same answer, we have

$$\langle f_{V_A} f_{V_B}\rangle = 2p(V, W) - 1 \tag{30}$$

so that the inequality may be rewritten

$$|p(V,W) - p(V,Z)| \leq p(W,Z). \tag{31}$$

Define $V(\theta)$ to be the subspace spanned by the vector

$$|\theta\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,. \tag{32}$$

The quantum model $(\mathcal{H}, \psi)$ assigns to these probabilities the values

$$p(V(\theta), V(\phi)) = \frac{1}{2} - \frac{\cos(2(\theta - \phi))}{2} \tag{33}$$

Taking $\theta = 0$, $\phi = \pi/8$, and $\chi = \pi/4$, we have

$$p(V(\theta), V(\phi)) = \frac{2 - \sqrt{2}}{4} \tag{34}$$

$$p(V(\theta), V(\chi)) = \frac{1}{2} \tag{35}$$

$$p(V(\phi), V(\chi)) = \frac{2 - \sqrt{2}}{4} \tag{36}$$

These do not satisfy the inequality 31. Therefore we have established that any hidden variable model that satisfies the anti-correlation property of $(\mathcal{H}, \psi)$ must satisfy a constraint (31) that is violated by $(\mathcal{H}, \psi)$. $\square$

4 References

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